6.8 epidemic models (ODE)

(viz Lec 2.10)

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Objectives:
1) Applying differential equations to modelling epidemics
2) Understand how to construct a model
3) See what can go wrong, during analysis.

Assumption 1: Each infected individual infects others at a rate β.

Model 1: (exp growth)

$$\frac{dI}{dt} = \beta I,$$ where $I(t) = \#$ of infected individuals at time t.

$$\Rightarrow \dot{I} - \beta I = 0$$

Char. eq: $\lambda - \beta = 0 \Rightarrow \lambda = \beta$

Ansatz: $I(t) = C e^{\beta t}$

$$\Rightarrow I(t) = I(0)\, e^{\beta t},$$ so exponential growth.

Fails to consider limited population size.

Assumption 2: There is a total fixed population size $N = I(t) + S(t)$, where S is the number of susceptible individuals.

Modified Assumption 1: The infection rate is proportional to β and to the fraction $\frac{S}{N}$ of susceptible.

SI model: (logistic growth)

$$\boxed{S} \xrightarrow{\frac{\beta I}{N}} \boxed{I}$$

$$N = S(t) + I(t)$$

$$\dot{S} = \frac{-\beta S I}{N}$$

$$\dot{I} = \frac{\beta S I}{N}$$

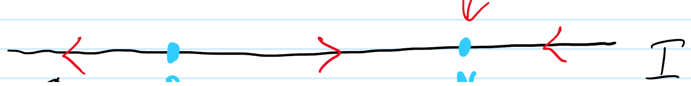
$$S = N - I$$

$$\dot{I} = \frac{\beta}{N}(N - I) I$$

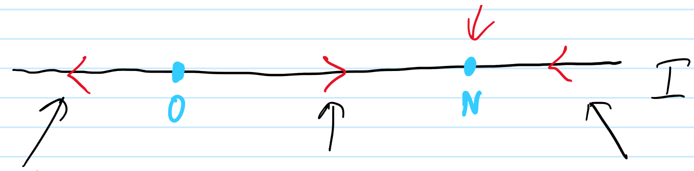
$$\dot{I} = \beta I \left(1 - \frac{I}{N}\right) = f(I)$$

Equilibria: $\bar{I} = 0, N$

globally asymptotically stable, for $I(0) > 0$.



for $I(0) > 0$.

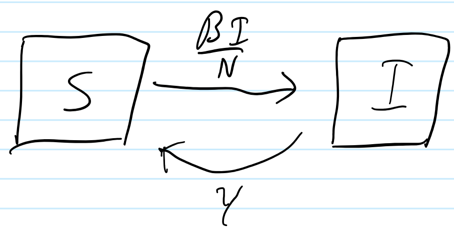


$f(I) = (-)(+) < 0$ if $I < 0$ $\qquad$ $f(I) = (+)(+) > 0$ if $0 < I < N$ $\qquad$ $f(I) = (+)(-) < 0$ if $I > N$.

irrelevant because it is not physical to have neg. infected individuals.

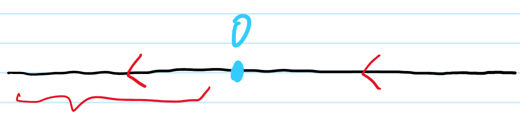
Assumption 3: Individuals recover at rate γ, but become susceptible again.

SIS model:



$$\left.\begin{aligned} N &= S(t) + I(t) \\ \dot{S} &= -\frac{\beta}{N} S I + \gamma I \\ \dot{I} &= \frac{\beta}{N} S I - \gamma I \end{aligned}\right\} \quad \begin{aligned} S &= N - I \\ \dot{I} &= \frac{\beta}{N}(N - I) I - \gamma I \\ \dot{I} &= \left(\beta - \frac{\beta I}{N} - \gamma\right) I \end{aligned}$$

Case 1: $\beta = \gamma$. Then $\dot{I} = -\frac{\beta}{N} I^2$, $\bar{I} = 0$, and 0 is globally asymp. stable for $I(0) \geq 0$.



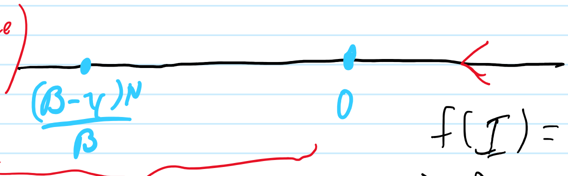
don't care about neg numbers

Case 2: $\beta \neq \gamma$ $\qquad$ $\dot{I} = (\beta - \gamma) I \left(1 - \frac{\beta}{(\beta - \gamma) N} \cdot I\right) = f(I)$

$\bar{I} = 0, \ \frac{(\beta - \gamma) N}{\beta}$ $\qquad$ $\left(\bar{S} = N, \ \frac{\gamma N}{\beta}\right)$

Case 2a: $\beta < \gamma$. Then $\bar{I} = 0$ is only pos eq.

(disease-free equilibrium)



$f(I) = (-)(+)(+) < 0$ if $I > 0$

$\Rightarrow 0$ is globally stable for $I(0) > 0$.

irrelevant

β irrelevant

$f(I) = (-)(+)(+) < 0$ if $I > 0$

$\Rightarrow$ 0 is globally stable for $I(0) > 0$.

Case 2b: $\beta > \gamma$. Then both $\bar{I} = 0$ and $\bar{I} = \frac{(\beta-\gamma)}{\beta} \cdot N$ are pos.

(endemic case)

$f(I) = (+)(+)(+) > 0$ $\quad$ $f(I) = (+)(+)(-) < 0$

Thus, $\bar{I} = \frac{(\beta-\gamma)}{\beta} \cdot N$ is globally asymp. stable for $I(0) > 0$.

$R_0 = \frac{\beta}{\gamma}$ is the basic reproduction number of the system because it is the number of secondary infections (β) caused during the infectious period $\frac{1}{\gamma}$.

$R_0 > 1 \Rightarrow \beta > \gamma \Rightarrow$ disease persists.

$R_0 < 1 \Rightarrow \beta < \gamma \Rightarrow$ disease dies out.

Modified assumption 3: Individuals recover at rate γ and become immune (Removed), i.e. no longer susceptible or infectious.

SIR model:

$$\boxed{S} \xrightarrow{\frac{\beta I}{N}} \boxed{I} \xrightarrow{\gamma} \boxed{R}$$

$N = S + I + R$

$$\begin{cases} \dot{S} = -\frac{\beta}{N} SI \\ \dot{I} = \frac{\beta}{N} SI - \gamma I = I\left(\frac{\beta}{N} S - \gamma\right) \\ \dot{R} = \gamma I \end{cases}$$

$R = N - S - I$, so can consider just S and I

$$\dot{S} = -\frac{\beta}{N} SI$$

$$\dot{I} = I\left(\frac{\beta}{N} S - \gamma\right)$$

S-nullcline: $\dot{S} = 0 = -\frac{\beta}{N} SI \Rightarrow I = 0$ or $S = 0$.

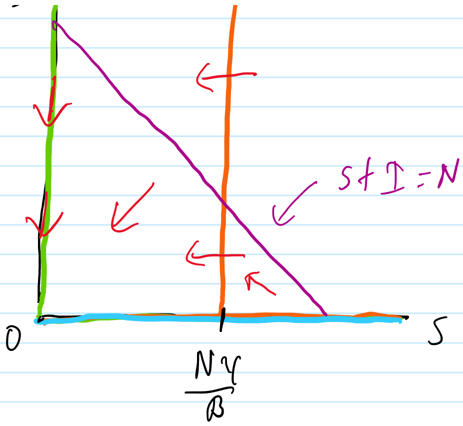
I-nullcline: $\dot{I} = 0 = I\left(\frac{\beta}{N} S - \gamma\right) \Rightarrow I = 0$ or $S = \frac{N}{\beta} \cdot \gamma$

Equilibria: $I = 0$, $S \in \mathbb{R}$.

Case 1: $\gamma < \beta$

$S + I < N$ because $S + I + R = N$.

Case 1: $\gamma < \beta$



$S+I<N$ because $S+I+R=N$.

If $S=0$, $\dot{I} = (+)(-) < 0$

If $S = \frac{N\gamma}{\beta}$, $\dot{S} = (-)(+) < 0$

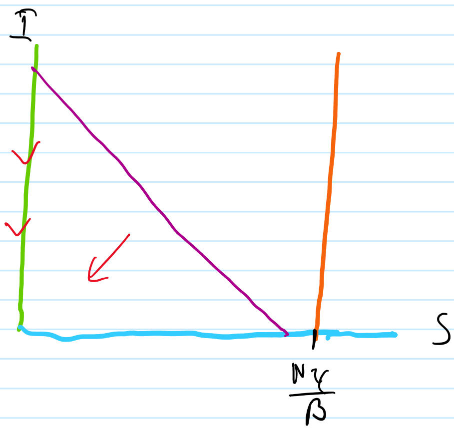
If $S < \frac{N\gamma}{\beta}$, $\dot{I} < 0$, $\dot{S} < 0$

If $S > \frac{N\gamma}{\beta}$, $\dot{I} > 0$, $\dot{S} < 0$

Note: If $S(0) > \frac{N\gamma}{\beta}$, then infections increase to a maximum at $S = \frac{N\gamma}{\beta}$ before decreasing, so epidemic happens.

If $S(0) < \frac{N\gamma}{\beta}$, then infections just die out to 0, at some pt on the S-axis.

Case 2: $\gamma > \beta$

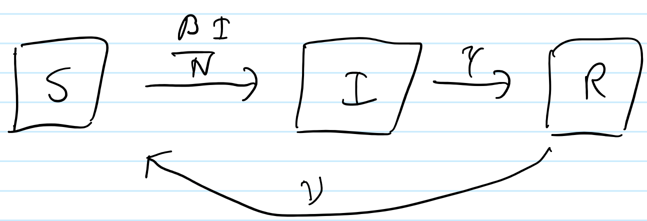


Because $S(0) \le N < \frac{N\gamma}{\beta}$, infections just die out and don't cause an epidemic.

$R_0 = \frac{\beta}{\gamma}$ is the basic reproduction number, and it determines if an epidemic is possible.

Assumption 4: Recovered individuals lose immunity at rate ν.

SIRS model:



$$\begin{cases} \dot{S} = -\frac{\beta}{N} SI + \nu R \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \\ \dot{R} = \gamma I - \nu R \end{cases}$$

Note $S(t) + I(t) + R(t) = N$

$\Rightarrow R = N - (S + I)$

so can substitute to get system of 2 equations.

$$\begin{cases} \dot{S} = -\frac{\beta}{N} SI + \nu(N - S - I) \\ \dot{I} = I\left(\frac{\beta}{N} S - \gamma\right) \end{cases}$$

Exercise 6.15: Use a phase plane analysis and determine equilibria and stability. What is R_0?